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STABILITY OF COUETTE FLOW IN GRANULAR MEDIA

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Introduction. Granular media are those which consist of a large number of solid particles (granules) with gas- and/or liquid-filled interstices between them. These media attract interest because they are abundant in nature and technological processes (snow and rock avalanches, mudflows, transportation of loose materials, mining of raw mineral materials, chemical industry, and powder metallurgy).

The fundamentals of modern knowledge of the mechanics of granular media are described in many papers (see, e.g., [1–3]). The first ideas were put forward by Bagnold in [1]. This paper contains the results of the first laboratory experiments and the estimations that describe the behavior of a moderately dense granular flow under shear. It is shown there that the pressure and the shear stress are proportional to the shear velocity squared if the angle of dynamic friction is not dependent on it. This quadratic dependence indicates that a granular medium behaves as a non-Newtonian fluid under certain conditions, whereas the stress and velocity of the shear are in linear relationship.

Two flow types of dry granular materials are distinguished, depending on the material density and the shear velocity:

(1) quasi-static flows which correspond to high densities and low velocities of the shear, in which the granules are always in close contact with each other, and the material behavior is fairly well described by the Mohr–Coulomb law;

(2) "grain-inertia" flows which correspond to the lower densities and higher velocities of the shear, in which there are certain gaps between the granules, and their interaction is caused by continuous collisions with each other. It is usually assumed that the mean free path of granules is not larger than their characteristic size. The limiting case of such a flow with a large mean free path is sometimes called a translational regime.

It is commonly known that the grain-inertia regime of granular flows can be studied using the laws of conservation of mass, momentum, and energy which are supplemented by some closing relations. Collisions between the granules are very important and, therefore, the energy of random motion of the granules should be taken into account, along with the energy of macroscopic motion. Similar to the gas, it can be called thermal energy, and the temperature can be introduced as a measure of intensity of the random motion of granules. Collisions are always inelastic in these media and, thus, the loss in energy of the granular material cannot be avoided in the energy balance.

The closing equations relate the pressure, viscosity, heat conductivity, and the decrease in the energy of random motion to the density and temperature of the medium. Appropriate relationships derived by analogy with the kinetic gas theory are presented in [4–6]. The continuum equations with these additional relations were used to solve a number of problems, including the problems of the linear stability of granular media which are either motionless or moving under conditions of constant velocity shear [7–11]. Certainly, granular media differ from gases in many respects, for example, by the size of granules in comparison with the size of molecules, and by the absence of large-distance attraction forces. Because of this, complicated structures associated with the development of the kinetic theories mentioned above are hardly fully justified. Therefore, the phenomenological approach [12] based on an analysis of the dimensions of physical quantities in the closing relations seems to us rather attractive. We shall describe briefly the essence of this approach.

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Pressure is the momentum variation per unit time per unit area. The momentum of a granule changes in each collision by approximately mv_T, where v_T is the root-mean-square (thermal) velocity of random motion of the granules. Multiplying this quantity by the mean collisional frequency v_T/s [s is the mean free path of the granules ($s \ll d$)] and dividing it by the characteristic size d squared, we obtain $p \simeq \rho v_T^2 d/s$ (ρ is the density of the material to be examined). The dynamic viscosity coefficient is the density multiplied by the area and divided by time, i.e., $\mu \simeq \rho v_T d^2/s$. The order of magnitude of the heat conductivity related to the exchange process of the mean energy of random motion of the granules during their collisions is determined in the same way as the dynamic viscosity. We obtain the rate of decrease in the energy of random motion similarly. Using this model, Haff [12] solved a number of problems of the mechanics of granular media: "cooling" of a uniform medium, determination of a steady state arising when a decrease in the thermal energy because of inelastic collisions is compensated by energy supply from outside, steady Couette flow with and without regard for the gravity force.

In the model proposed in [12], the granules are assumed to be absolutely rigid, and the time of their contact during collisions is assumed to infinitesimal. Hwang and Hutter generalized this model [13] with allowance for the compressibility of the medium and the deformation of granules during collisions, which leads to a finite time of contact of the granules. In other words, the model acquires the variable density $\rho(d/(d+s))^3$ and the time of contact of the granules is $t_c = \alpha d/c$, where $c = (E/\rho)^{1/2}$ (E is the Young's elasticity modulus) and α is a nondimensional parameter of the order of unity. This contact time is the time necessary for a plane deformation wave to cross the granule's diameter back and forth. The total time between the collisions is then determined as $t_e = t_f + t_c$ ($t_f = s/v_T$ is the mean free time of the granules), and the pressure, transfer coefficients, and loss of thermal energy are obtained in [13] by substituting t_e for t_f in appropriate quantities from [12]. It is shown that the account of the finite contact time slows down the relaxation processes.

In the present work, we study the dispersion properties of a granular medium described by this model [13].

1. Governing Equations. With a granular medium considered as a continuum, we write the equations for the density, macroscopic velocity, and energy of random motion or temperature of the granules:

$$\frac{d\rho}{dt} = -\rho\frac{\partial u_i}{\partial x_i}, \quad \rho\frac{du_i}{dt} = -\frac{\partial P_{ij}}{\partial x_j}, \qquad \frac{3}{2}\rho\frac{dT}{dt} = \frac{\partial}{\partial x_i}\Big(æ\frac{\partial T}{\partial x_i}\Big) - P_{ij}\frac{\partial u_i}{\partial x_j} - I. \tag{1.1}$$

The stress tensor has the form

$$P_{ij} = \Big[p - \Big(\zeta - \frac{2\mu}{3}\Big)\frac{\partial u_k}{\partial x_k}\Big]\delta_{ij} - \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \tag{1.2}$$

where $d/dt = \partial/\partial t + u_i\partial/\partial x_i$, $\mathbf{u} = (u, v, w)$ is the macroscopic velocity, p is the pressure, $T \equiv v_T^2$ is the temperature of the granules, ζ is the coefficient of bulk (second) viscosity, $æ$ is the heat conductivity, and I is the loss of thermal energy in inelastic collisions of the granules per unit volume per unit time. As elsewhere, the twice repeated subscripts denote summation.

Equations (1.1) should be supplemented by closing equations which relate the pressure, transfer coefficients, and loss of thermal energy to the density, temperature, and, possibly, some other functions which characterize the state of the medium. Precisely the closing equations specify the model to be used for studying the processes in granular media.

As mentioned in the Introduction, we use phenomenological relationships for the pressure, viscosity and diffusion coefficients, and rate of decrease in the thermal energy of the granules, which are based on a dimensional analysis of the physical quantities in the governing equations. If we assume, as in [13], that the density depends on the mean free path and the contact time of the granules during their collisions is finite, the desired formulas, with accuracy to nondimensional coefficients of the order of unity, have the form

$$\begin{gathered} \rho = m(d+s)^{-3}, \quad p = mv_T(d+s)^{-2}t_e^{-1}, \\ \mu, \zeta, æ = m(d+s)^{-1}t_e^{-1}, \quad I = m(1-e^2)v_T^2(d+s)^{-3}t_e^{-1}. \end{gathered} \tag{1.3}$$

Here $m = \rho_p d^3$, $t_e = t_f + t_c$, $t_f = s/v_T$, $t_c = \alpha d/c$, and e is the coefficient of restitution after inelastic collisions of the granules. For absolutely rigid granules ($E, c \to \infty$), formulas (1.3) are transformed to the formulas from [12]. Thus, the pressure, momentum and heat-transfer coefficients, and decrease in the energy of random motion of the granules are functions of the mean free path and temperature, and the physical properties of the substance enter these characteristics in terms of the Young's modulus and the coefficient of restitution. The functions p, μ, ζ, æ, and I decrease with increasing mean free path, whereas their dependence on temperature is more complicated.

2. Formulation of the Problem of the Dispersion Properties of a Medium. As in the usual procedure, for analysis of the dispersion properties of any medium, the governing equations are linearized with respect to a certain equilibrium state disturbed by small perturbations in the desired functions. If the solution of this linearized system for perturbations is represented as a superposition of plane waves, the existence condition of a nontrivial solution leads to the dispersion equation $\omega = \omega(k)$ which determines the frequency of plane waves versus their wavenumber. An analysis of the roots of this equation allows us to draw a conclusion on the stability or instability of the initial equilibrium state. If the frequency is complex for real wavenumbers, then $\exp i(\mathbf{kr} - \omega t) = \exp(\mathrm{Im}\,\omega t)\exp i(\mathbf{kr} - \mathrm{Re}\,\omega t)$, and we easily see that the positive sign of $\mathrm{Im}\,\omega \equiv \gamma$ denotes an increase in the perturbation amplitude, while a negative sign denotes a decrease in the amplitude. The equilibrium state of the medium is unstable in the first case and stable in the latter case.

Collisions between the granules are inelastic, and, therefore, a state that corresponds to a macroscopically motionless medium is not an equilibrium state. Indeed, in the simplest case of a uniform medium the third equation in (1.1) reduces to the equation $dT/dt = -2I/3\rho$. Since I is positive, the temperature decreases in time, asymptotically approaching zero, and all transfer coefficients also tend to zero. Hence, a nontrivial situation is possible only if there is mechanism of compensation for the energy of random motion of the granules. This arises, in particular, in shear (Couette) flows where the dissipation of the thermal energy of the granules is compensated by the work of external forces which ensure the existence of a shear flow.

We consider a granular-medium flow with constant shear between two plates moving parallel to each other with different velocities. In the steady state, the macroscopic velocity has one component depending on the coordinate y, i.e., $\mathbf{u}_0 = (u_0(y), 0, 0)$. The continuity equation and the z component of the motion equation are satisfied identically, and the x and y components of the motion equation and the temperature equation yield

$$\frac{dp_0}{dy} = 0, \quad \frac{d}{dy}\left(\mu_0 \frac{du_0}{dy}\right) = 0, \quad \frac{d}{dy}\left(æ_0 \frac{dT_0}{dy}\right) + \mu_0 \left(\frac{du_0}{dy}\right)^2 = I_0. \tag{2.1}$$

The simplest solution of Eqs. (2.1) corresponding to a flow with constant shear is

$$\rho_0 = \mathrm{const}, \quad \frac{du_0}{dy} = \Gamma = \mathrm{const}, \quad p_0 = m v_T (d + s_0)^{-2} t_e^{-1}, \quad \mu_0, \zeta_0, æ_0 = m(d + s_0)^{-1} t_e^{-1},$$
$$v_T^2 = T_0 = (1 - e^2)^{-1}(d + s_0)^2 \Gamma^2, \qquad I_0 = m(1 - e^2) v_T^2 (d + s_0)^{-3} t_e^{-1}. \tag{2.2}$$

Here the shear Γ is a constant quantity. Since $\rho = m(d + s_0)^{-3}$, the uniform density denotes a constant mean free path.

We disturb the steady state (2.2) of the medium by adding small perturbations to all functions, i.e., we set $f(\mathbf{r}, t) = f_0 + f'(\mathbf{r}, t)$ and $f' \ll f_0$, substitute these relations into Eqs. (1.1)–(1.3), and retain all first-order quantities. Henceforth, the primes are omitted. The coefficient u_0 in the resultant system is a function of the y coordinate, which complicates the analysis. Therefore, we shall consider perturbations that propagate perpendicular to the shear plane x, y. In other words, we confine ourselves to perturbations with the wave vector $\mathbf{k} = (0, 0, k)$, representing them in the form $\sum A_k \exp i(kz - \omega t)$. Since $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$, this system of linear equations reduces to two subsystems, one of which contains the functions u and v, and the other has the functions ρ, w, and T:

$$\rho_0(u_t + \Gamma v) = \mu_0 u_{zz}, \quad \rho_0 v_t = \mu_0 v_{zz}; \tag{2.3}$$

$$\rho_t = -\rho_0 w_z, \quad \rho_0 w_t = -p_z + (\zeta_0 + 4\mu_0/3) w_{zz}, \quad (3/2)\rho_0 T_t = æ_0 T_{zz} - p_0 w_z + \Gamma^2 \mu - I. \tag{2.4}$$

We seek a solution in the form of plane waves, and, thus, assume that $\partial/\partial t = -i\omega$ and $\partial/\partial z = ik$ in Eqs. (2.3) and (2.4).

From system (2.3), we obtain $v = 0$ and $\omega = -i(\mu_0/\rho_0)k^2$, which corresponds to a decaying nonpropagating harmonic with the decrement $\gamma = -\mu_0 k^2/\rho_0$, which has only the velocity component u parallel to the steady flow velocity.

System (2.4) describes longitudinal waves (relative to the wave vector of perturbations), in which the density, velocity component along the wave vector, and temperature vary. The continuity equation leads to the following relationship between the velocity and density (or mean free path) of perturbations: $\rho = k\rho_0 w/\omega$ and $w = -3\omega s(d + s_0)^{-1}k^{-1}$. The pressure, transfer coefficients, and rate of decrease in the energy of random motion of the granules (1.3) are functions of the mean free path and temperature, and we, therefore, write perturbations in the form

$$p, \mu, I = (p_1, \mu_1, I_1)s + (p_2, \mu_2, I_2)T, \qquad p_1, \mu_1, I_1 = \frac{\partial(p, \mu, I)}{\partial s}, \qquad p_2, \mu_2, I_2 = \frac{\partial(p, \mu, I)}{\partial T}.$$

Substituting these relations into (2.4) and equating the determinant to zero, we obtain the cubic equation

$$\gamma^3 + (ak^2 - b)\gamma^2 + k^2(c_1 k^2 - d_1)\gamma - k^2(e_1 k^2 + f_1) = 0, \tag{2.5}$$

where

$$\begin{gathered}\gamma = -i\omega, \quad a = (2/3\rho_0)(3\rho_0 v_0 + æ_0), \quad \nu_0 = (4\mu_0/3 + \zeta_0)/\rho_0, \quad c_1 = 2æ_0\nu_0/3\rho_0, \\ b = (2/3\rho_0)(\mu_2\Gamma^2 - I_2), \quad d_1 = (2/3\rho_0)(p_1(d + s_0)/2 - p_0 p_2/\rho_0 + 3\rho_0\nu_0 b), \\ e_1 = 2æ_0 p_1(d + s_0)/9\rho_0^2, \quad f_1 = 2(d + s_0)(p_2(\mu_1\Gamma^2 - I_1) - p_1(\mu_2\Gamma^2 - I_2))/9\rho_0^2;\end{gathered} \tag{2.6}$$

$$\begin{gathered}p_1 = -m(d + 3s_0 + 2v_T t_c)(d + s_0)^{-3}, \quad p_2 = m(t_f + t_c/2)v_T^{-1}t_e^{-2}(d + s_0)^{-2}, \\ \mu_1 = -m(d + 2s_0 + v_T t_c)v_T^{-1}t_e^{-2}(d + s_0)^{-2}, \quad \mu_2 = (1/2)m s_0 v_T^{-3}t_e^{-2}(d + s_0)^{-1}, \\ I_1 = -m(1 - e^2)v_T(d + 4s_0 + 3v_T t_c)(d + s_0)^{-4}t_e^{-2}, \\ I_2 = (1/2)m(1 - e^2)(3t_f + 2t_c)(d + s_0)^{-3}t_e^{-2}.\end{gathered} \tag{2.7}$$

The real parts of the roots γ in the dispersion equation (2.5) correspond to the growth rate ($\gamma > 0$) or decrement ($\gamma < 0$) of the perturbation amplitude in time, whereas the imaginary parts correspond to the perturbation frequency. The solution of this equation allows us to find the regions of stability or instability of the granular flow considered, depending on the parameters of the medium and the perturbation wavelengths or wavenumbers.

3. Analysis of the Solutions of the Dispersion Equation. Since the problem of the stability of a granular flow is considered within the framework of continuum mechanics, the minimum wavelength of the perturbations examined should be substantially larger than the granule diameter or, more exactly, than the characteristic size equal to the sum of the granule's diameter and the mean free path. Thus, $\lambda_{\min} = A(d + s_0)$ ($A \gg 1$), and the maximum wavenumber that fits the limits of applicability of the continuum model is $k_{\max} = 2\pi A^{-1}(d + s_0)^{-1}$.

Since the dispersion equation is a cubic one, it has three roots for three harmonics that can exist in the medium considered. First of all, we shall study the behavior of long-wave perturbations with small wavenumbers ($k \ll 1$) for which an analytical solution can be obtained. For doing so, we present the growth rate in the form $\gamma = \gamma_0 + k\gamma_1 + k^2\gamma_2 + \ldots$. Confining ourselves to second-order quantities inclusive and substituting the expansions for γ into the dispersion equation (2.6), we collect separately the terms of zeroth, first, and second orders. In the zeroth order, we have $\gamma_0^3 - b\gamma_0^2 = 0$ from which it follows that $\gamma_0^{(1)} = b$ and $\gamma_0^{(2,3)} = 0$, where the superscripts denote the perturbation-harmonic number. In the first order with respect to the wavenumber, we have $\gamma_0\gamma_1(\gamma_0 - 2b/3) = 0$. Substituting $\gamma_0^{(1)}$, we find $\gamma_1^{(1)} = 0$, and to find the two

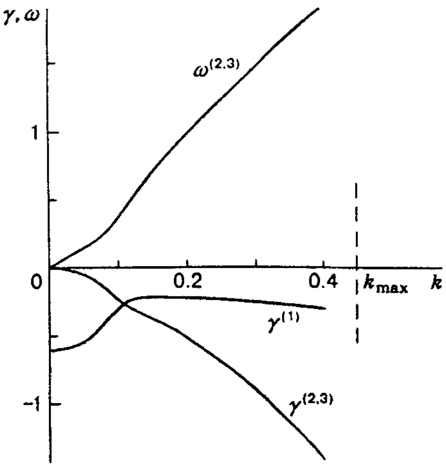

Fig. 1

remaining roots $\gamma_1^{(2,3)}$, we use the second approximation. In the second order with respect to the wavenumber, we obtain the equation

$$3\gamma_0(\gamma_1^2 + \gamma_0\gamma_2) + a\gamma_0^2 - b(\gamma_1^2 + 2\gamma_0\gamma_1) - d_1\gamma_0 - f_1 = 0.$$

Substituting $\gamma_0^{(2,3)} = 0$ and $\gamma_0^{(1)} = b$ into it, we obtain

$$\gamma_1^{(2,3)} = \pm i\sqrt{f_1/b}, \qquad \gamma_2^{(1)} = (d_1 b + f_1 - ab^2)/b^2.$$

Coming back to notation (2.6) and (2.7), we write the roots of the dispersion equation in the long-wave range. For the first harmonic, we write

$$\gamma^{(1)} = -2(1 - e^2)/3t_e + O(k^3), \tag{3.1}$$

from which it follows that the first harmonic is nonpropagating ($\omega = 0$) and decaying ($\gamma < 0$). If the mean collision time of contact of the granules t_c is much smaller than the mean free time t_f, the decrement modulus increases with decreasing coefficient of restitution as $(1 - e^2)^{1/2}$, and the decrement of the first harmonic is directly proportional to the thermal velocity of motion of the granules and inversely proportional to the mean free path.

For the second and third harmonics, we obtain

$$\gamma^{(2,3)} = \pm ik(d + s_0)^{1/2}(d + s_0 + v_T t_c)^{1/2}/(\sqrt{3}t_e) + O(k^2).$$

It follows from this formula that the growth rates of the second and third harmonics in the long-wave range are purely imaginary. Since $\gamma = -i\omega$ by definition, the frequencies are real, and these harmonics are nondecaying plane waves propagating in the opposite directions with phase velocity

$$v_{\text{ph}} = \omega^{(2,3)}/k = 3^{-1/2} v_T (d + s_0) s_0^{-1} (1 + t_c/t_f)^{-1} (1 + s_0 t_c/t_f\,(d + s_0))^{1/2}.$$

The ratio t_c/t_f being given, the phase velocity increases with increasing thermal velocity of the granules and decreases with a decrease in the coefficient of restitution as $v_{\text{ph}} \sim (1-e^2)^{-1/2}$, since $v_T = (1-e^2)^{-1/2}(d+s_0)\Gamma$.

Thus, the Couette granular flow examined is neutrally stable with respect to long-wave perturbations which propagate perpendicular to the shear plane.

To study the stability of a granular flow within the entire range of wavenumber variation ($0 \leqslant k \leqslant k_{\max}$), we found the roots of Eq. (2.6) using the Cardano formulas for various parameters of the medium. Figure 1 shows the decrements and frequencies of nonpropagating and propagating harmonics as functions of wavenumber for the mean free path $s_0 = 0.05d$, $e = 0.9$, and $t_c = 0$. The first harmonic decays most rapidly in the long-wave range $kd < 0.05$. As the wavenumber increases, the decrement modulus decreases and reaches a minimum with $kd \approx 0.17$ ($\lambda = 37d$). A further increase in the wavenumber leads to a nearly

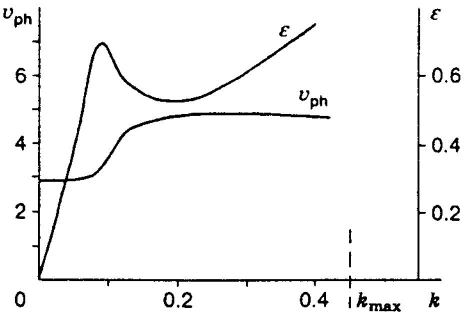

Fig. 2

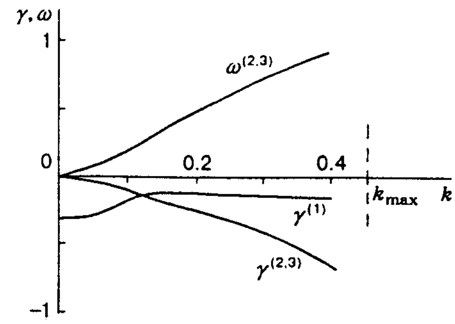

Fig. 3

linear increase in the decrement modulus by approximately 1.5 times (for k_{max}) compared with its minimum value. The frequency of propagating harmonics for $0 \leqslant kd \leqslant 0.06$ ($\lambda \geqslant 100d$) increases linearly with increasing wavenumber, and the phase velocity in this long-wave region is not dependent on the wavenumber, i.e., the waves do not possess dispersion. For $kd > 0.06$, the function $\omega^{(2,3)} = \omega(k)$ is not linear, which corresponds to positive dispersion for which the phase velocity increases with increasing wavenumbers. This means that the shorter waves propagate faster than the longer ones. In accordance with the general theory of dispersion waves, finite-amplitude rarefaction waves can exist in this region. The frequency grows with subsequent increase in the wavenumbers, but the increase in the velocity of propagating harmonics becomes slower, and the phase velocity is almost constant, approximately 1.7 times larger than for $k \to 0$, in the region $0.2 \leqslant kd \leqslant 0.35$.

The propagating harmonics decay in time, their decrement modulus, which is very small in the long-wave range, increases monotonically up to the limit k_{max}, and the wave-attenuation coefficient $\varepsilon = \gamma/\omega$ increases linearly as the wavenumbers vary from $kd \geqslant 0$ to $kd \approx 0.06$ and reaches a local maximum ($\varepsilon = 0.69$) at $kd \approx 0.09$. As the wavenumber increases, the attenuation coefficient passes through a minimum ($\varepsilon_{min} \approx 0.53$) at $kd \approx 0.18$ and then monotonically increases up to 0.83 at $kd = (kd)_{max} = 0.45$. The phase velocity and the attenuation coefficient of propagating harmonics are shown in Fig. 2 as functions of the wavenumber.

Taking into account the finite contact time t_c does not vary qualitatively the character of perturbation evolution. Figure 3 plots $\gamma^{(1)}$, $\gamma^{(2,3)}$, and $\omega^{(2,3)}$ versus the wavenumber for $t_c = t_f$. We see that the decrement modulus of the first harmonic in the long-wave range is twice as small as in the case of absolutely rigid collisions ($t_c = 0$). The same decrease is observed for the decrement and frequency of the propagating modes.

An increase of the mean free path s_0 for $t_c = 0$ is qualitatively similar to an increase in the degree of deformation of the granules in collisions: the decrements, frequencies, and phase velocities of the harmonics decrease. For $s_0 = 0.5d$ and $t_c = 0$, the phase velocity v_{ph}, thus, varies from 0.6 as $k \to 0$ to ≈ 1.1 in the medium range of wavenumbers ($0.15 \leqslant kd \leqslant 0.30$), and then decreases to 0.95 for $k = k_{max}$.

Thus, a granular constant-shear flow described by the model proposed in [13] is stable relative to small perturbations within the entire range of wavelengths of the waves propagating across the shear plane.

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